

VAPNIK-CHERVONENKIS DENSITY ON INDISCERNIBLE SEQUENCES, STABILITY, AND THE MAXIMUM PROPERTY

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ABSTRACT. This paper presents some finite combinatorics of set systems with applications to model theory, particularly the study of dependent theories. There are two main results. First, we give a way of producing lower bounds on VC_{ind} -density, and use it to compute the exact VC_{ind} -density of polynomial inequalities, and a variety of geometric set families. The main technical tool used is the notion of a maximum set system, which we juxtapose to indiscernibles. In the second part of the paper we give a maximum set system analogue to Shelah's characterization of stability using indiscernible sequences.

1. INTRODUCTION

In the recent past there have been a number of papers relating various measures of the combinatorial structure of NIP theories to one another [1, 5, 7]. One fact which emerged from this is the close relation of dp-rank and VC-density restricted to indiscernible sequences. Guingona and Hill have used the term VC_{ind} -density to describe VC-density restricted to indiscernibles. At the end of their paper [5], Guingona and Hill ask if there is a useful characterization of when a formula has VC_{ind} -density equal to VC-density. We offer such a characterization below (Corollary 3.5), and use it to compute the exact VC_{ind} -density of certain formulas.

A separate goal of this paper is to show how maximum set systems can in many cases be used as more accessible surrogates for indiscernible sequences. To this end we translate Shelah's well-known characterization of stability in terms of indiscernible sets to a version involving maximum set systems.

2. NOTATION

Let there be a fixed complete theory T , with a large saturated model $\mathfrak{M} = \langle M, \dots \rangle$. All sets and models, unless otherwise stated, are assumed to be elementarily embedded in the model \mathfrak{M} . We write formulas in partitioned form $\varphi(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = \langle x_1, \dots, x_l \rangle$, and $\mathbf{y} = \langle y_1, \dots, y_k \rangle$.

For $A \subseteq M^{|\mathbf{y}|}$ and $\mathbf{b} \in M^{|\mathbf{x}|}$,

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$$\varphi(\mathbf{b}, A) := \{\mathbf{a} \in A : \models \varphi(\mathbf{b}, \mathbf{a})\}$$

We use $S_\varphi(A)$ for $A \subseteq M^{|y|}$ to denote the set of φ -types over A , where a φ -type over A is a maximal consistent set of the form $\{\pm\varphi(\mathbf{x}, \mathbf{a}) : \mathbf{a} \in M^{|y|}\}$. We let $S_\varphi(A)|_B = \{tp_\varphi(\mathbf{b}/A) : \mathbf{b} \in B\}$ when $B \subseteq M^{|x|}$. We sometimes identify $S_\varphi(A)|_B$ and $\{C \subseteq A : C = \varphi(\mathbf{b}, A), \text{ for some } \mathbf{b} \in B\}$, as in the following definition. For a set X , let $\mathcal{P}(X)$ denote the power set of X .

Definition 2.1. *If $A \subseteq M^{|y|}$ and $\mathcal{C} \subseteq \mathcal{P}(A)$, we will say that $\varphi(\mathbf{x}, \mathbf{y})$ traces \mathcal{C} if for some $B \subseteq M^{|x|}$, $S_\varphi(A)|_B = \mathcal{C}$.*

We now give some purely combinatorial definitions. For the rest of this section suppose X is a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. For $A \subseteq X$, let $\mathcal{C}|^A = \{C \cap A : C \in \mathcal{C}\}$. Say that \mathcal{C} *shatters* A if $\mathcal{C}|^A = \mathcal{P}(A)$.

Definition 2.2. *The Vapnik-Chervonenkis (VC) dimension of \mathcal{C} , denoted $VC(\mathcal{C})$, is $|A|$ where $A \subseteq X$ is of maximum cardinality such that \mathcal{C} shatters A .*

If the VC-dimension of \mathcal{C} does not exist, we write $VC(\mathcal{C}) = \infty$. The VC dimension was first considered in [16] and was introduced into model theory by Laskowski [11]. The following notion of a maximum VC family was defined by Welzl [17].

Definition 2.3. *Say that \mathcal{C} is d -maximum for $d \in \omega$ if both of the following hold.*

- (1) $VC(\mathcal{C}) = d$
- (2) *For any $A \subseteq X$, finite, $|\mathcal{C}|^A| = \binom{|A|}{\leq d}$.*

Here $\binom{n}{\leq k}$ is shorthand for $\sum_{i=0}^k \binom{n}{i}$.

Lemma 2.1 (Sauer's Lemma [13, 14]). *If \mathcal{C} has $VC(\mathcal{C}) = d$, then for any finite $A \subseteq X$,*

$$|\mathcal{C}|^A| \leq \binom{|A|}{\leq d}$$

Thus a d -maximum set system is “extremal” among set systems of VC-dimension d . These set systems are highly structured, and well-understood [4, 10]. There are several examples that arise from natural algebraic situations. In fact it is conjectured that all d -maximum set systems arise from (or embed naturally in) arrangements of half-spaces, either in a euclidean or hyperbolic space [12].

It is easy to see that if \mathcal{C} is d -maximum on X , and $A \subseteq X$ has $|A| = d+1$, then

$$\mathcal{C}|^A = \mathcal{P}(A) \setminus \{C\}$$

for some $C \subseteq A$. Floyd [4] calls such a C the *forbidden label* of \mathcal{C} on A .

Let $[X]^m := \{A \subseteq X : |A| = m\}$. For a fixed d -maximum $\mathcal{C} \subseteq \mathcal{P}(X)$, associate each $A \in [X]^{d+1}$ with the forbidden label $C_A \subseteq A$, where $\mathcal{C}|^A = \mathcal{P}(A) \setminus \{C_A\}$.

Floyd proves the following.

Proposition 2.2. *On a finite domain X , any d -maximum set system \mathcal{C} is characterized by its forbidden labels, in the sense that $\forall B \subseteq X$,*

$$B \in \mathcal{C} \iff \forall A \in [X]^{d+1} (B \cap A) \neq C_A$$

Proof. Left to right is obvious. For right to left, let B satisfy the given conditions. Then $\mathcal{C} \cup \{B\}$ shatters no sets not shattered by \mathcal{C} . By Sauer's Lemma, B must already be in \mathcal{C} . \square

We now define the notion of a forbidden code, which is essentially the “form” of a forbidden label, when an ordering is present.¹

Let $\mathfrak{L}_{\mathcal{C}}(X) = \{C_A \in [X]^{\leq d+1} : A \in [X]^{d+1}\}$ denote the forbidden labels of \mathcal{C} on X . Let $<$ be a fixed but arbitrary linear order on X . For each $C_A \in \mathfrak{L}_{\mathcal{C}}(X)$, let $\overline{C_A} = \langle t_0, \dots, t_d \rangle$, where each $t_i \in \{0, 1\}$, and $t_i = 1$ if and only if the i th element of A is in C_A . Define $\overline{\mathfrak{L}_{\mathcal{C}}(X)} = \{\overline{C_A} : C_A \in \mathfrak{L}_{\mathcal{C}}(X)\}$.

We will refer to $\overline{\mathfrak{L}_{\mathcal{C}}(X)}$ as the set of *forbidden codes* on X for \mathcal{C} , with respect to $<$.

There is a natural way in which forbidden codes can serve as combinatorial invariants for finite unions of points and $<$ -convex subsets in X . To see this, suppose $(X, <)$ is a dense and complete linear order, and $B \subseteq X$ is a finite union of convex subsets. Let d be the number of boundary points of B . We can imagine that B is defined by some $L_{<} = \{<\}$ formula $\psi(x, c_1, \dots, c_d)$ with $c_1 < \dots < c_d \in X$. Let $\mathcal{F}(B) = \{\psi(X, c'_1, \dots, c'_d) : c'_1 < \dots < c'_d \in X\}$. Intuitively the elements of $\mathcal{F}(B) \subseteq \mathcal{P}(X)$ are the “homeomorphic images” of B in $(X, <)$. In [8] we show that $\mathcal{F}(B)$ is finitely characterized by some $\eta \in 2^{d+1}$. Recall that being *finitely characterized* by η means that for any finite $X_0 \subseteq X$, and any $A \subseteq X_0$, there is $C \in \mathcal{F}(B)$ such that $C \cap X_0 = A$ if and only if there are not $a_0 < \dots < a_d$ in X_0 such that $a_i \in A$ if and only if $\eta(i) = 1$.

Define the *genus* of B , denoted $\mathbb{G}(B)$, to be the $\eta \in 2^{d+1}$ that finitely characterizes $\mathcal{F}(B)$. Equivalently, define $\mathbb{G}(B)$ to be any $\eta \in 2^{d+1}$ such that there are no $a_0 < \dots < a_d$ in X such that $a_i \in B$ if and only if $\eta(i) = 1$, for all $i = 0, \dots, d$.

We will take it to be established (see [8]) that such an η exists, and is unique, as well as the further fact that for any $\eta \in 2^{<\omega}$ there is some $B \subseteq X$ such that $\mathbb{G}(B) = \eta$. Simple rules for computing genus are given in Table 1.

To give an example of applying the table, suppose $X = \mathbb{R}$, and $<$ is the usual ordering. Then the genus the point $\{0\}$ is $\langle 11 \rangle$, and the genus of the interval $(0, 1)$ is $\langle 101 \rangle$. Conversely, to consider all subsets of \mathbb{R} with genus

¹In [8] we did not distinguish between forbidden codes and labels, but referred to both using the same term.

code	translation
$\langle 1 \dots \rangle$	do nothing
$\langle 0 \dots \rangle$	$(-\infty, \dots$
$\langle \dots 0, 0 \dots \rangle$	remove point
$\langle \dots 0, 1 \dots \rangle$	end interval
$\langle \dots 1, 0 \dots \rangle$	begin interval
$\langle \dots 1, 1 \dots \rangle$	add point
$\langle \dots 0 \rangle$	$\dots, \infty)$
$\langle \dots 1 \rangle$	do nothing

TABLE 1. A key for assigning forbidden codes to unions of convex sets.

$\langle 11 \rangle$, let \mathcal{C} be all the singletons. Similarly, the collection of all subsets of genus $\langle 101 \rangle$ is exactly the set of all infinite convex subsets which are not coinitial or cofinal.

The assumption that $(X, <)$ is complete was made to give a clear presentation of the genus concept, and is sufficient for this paper. One can, however, define the genus of $B \subseteq X$ on other orders by considering the shortest $\eta \in 2^{<\omega}$ which B does not induce, sidestepping the issue of boundary points.

The link between genus and forbidden codes is given in the following theorem.

Theorem 2.3.

Suppose that $(X, <)$ is a complete dense linear order without endpoints. If $\eta \in 2^{d+1}$, and $\mathcal{C} = \{C \subseteq X : \mathbb{G}(C) = \eta\}$, then \mathcal{C} is d -maximum on X and $\mathfrak{L}_{\mathcal{C}}(X) = \{\eta\}$.

Proof. We provide a sketch of the proof that \mathcal{C} is d -maximum, which is very similar to the well-known proof that unions of intervals are maximum.

Let $X_0 \subseteq X$ be finite, and $\mathcal{C} = \{C \subseteq X : \mathbb{G}(C) = \eta\}$. Let $a := \max\{X_0\}$ and $X_0^a = X_0 \setminus a$. Define $\mathcal{C}^a = \{C \in \mathcal{C}|^{X_0^a} : C \cup \{a\} \in \mathcal{C}|^{X_0} \text{ \& } C \in \mathcal{C}|^{X_0^a}\}$. Show, by induction on $|X_0|$ and d , that \mathcal{C}^a is $d-1$ maximum and $\mathcal{C}|^{X_0^a}$ is d -maximum. Then show $|\mathcal{C}|^{X_0} = |\mathcal{C}^a| + |\mathcal{C}|^{X_0^a}$, and that, by Pascal's identity, $|\mathcal{C}|^{X_0} = \binom{|X_0|}{\leq d}$.

□

3. RESULTS

3.1. VC_m and VC_{ind}-density. We now apply the above to achieve our results. Recall the following definitions.

Definition 3.1. *A formula $\varphi(\mathbf{x}, \mathbf{y})$ has VC-density $\leq r$ for $r \in \mathbb{R}$ if there is $K \in \omega$ such that for every finite $A \subseteq M^{\mathbf{y}}$, $|S_{\varphi}(A)| < K \cdot |A|^r$. We denote this by $\text{VCd}(\varphi) \leq r$.*

Definition 3.2. A formula $\varphi(\mathbf{x}, \mathbf{y})$ has VC_{ind} -density $\leq r$ for $r \in \mathbb{R}$ if there is $K \in \omega$ such that for every finite and indiscernible $A \subseteq M^{|\mathbf{y}|}$, $|S_\varphi(A)| < K \cdot |A|^r$. We denote this by $\text{VCd}_{\text{ind}}(\varphi) \leq r$.

The study of VC-density has emerged several times in model theory. See [1] for historical remarks.

There has been some study of the fact that frequently $\text{VCd}(\varphi)$ is bounded by a simple (and uniform) function of $|\mathbf{x}|$ [1, 9]. When this is true, it justifies the heuristic practice of “parameter counting” to guess the complexity of set systems. Guingona and Hill showed that in a dp -minimal theory $\text{VCd}_{\text{ind}}(\varphi) \leq |\mathbf{x}|$. Thus there is interest in bounding $\text{VCd}(\varphi)$ by a function of $\text{VCd}_{\text{ind}}(\varphi)$ (obviously $\text{VCd}(\varphi) \geq \text{VCd}_{\text{ind}}(\varphi)$.) This may not be possible in general, but we now show a practicable route to achieving it for a given formula.

Definition 3.3.

$$\text{Tr}(\varphi, M^{|\mathbf{y}|}) := \{\mathcal{C} : \exists A \subseteq M^{|\mathbf{y}|}, \mathcal{C} \subseteq \mathcal{P}(A), \varphi \text{ traces } \mathcal{C}\}$$

The following is easy, but interesting, because the collection of d -maximum set systems would seem, *a priori*, to be very diverse. It also informs Definition 3.4.

Lemma 3.1. For each $d \in \omega$, there is some $\mathcal{C} \in \text{Tr}(\varphi, M^{|\mathbf{y}|})$ such that $|\mathcal{C}| \geq \aleph_0$ and \mathcal{C} is d -maximum if and only if for each $n \in \omega$, $n \geq d$, there is $\mathcal{C}_n \in \text{Tr}(\varphi, M^{|\mathbf{y}|})$ such that $|\bigcup \mathcal{C}_n| = n$ and \mathcal{C}_n is d -maximum.

Proof. For the right to left direction, it is easily seen that the property of being d -maximum is elementary. Apply the compactness theorem and the saturation of \mathfrak{M} . For left to right, note that if \mathcal{C} is d -maximum and $X' \subseteq X$ with $|X'| = n$, $n \geq d$, then $\mathcal{C}|^{X'}$ is d -maximum. \square

Definition 3.4.

$$\text{VCm}(\varphi) := \max\{d \in \omega : \exists \mathcal{C} \in \text{Tr}(\varphi, M^{|\mathbf{y}|}), |\mathcal{C}| \geq \aleph_0, \mathcal{C} \text{ is } d\text{-maximum}\}$$

If $\langle \mathbf{a}_i \rangle_{i \in I}$ is a sequence of indiscernibles and $(I, <)$ is a complete and dense linear order without endpoints, for $B \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$, define $\mathbb{G}(B)$ to be the genus of $\{i \in I : \mathbf{a}_i \in B\} \subseteq I$. With these assumptions, for $m \in \omega$, define $P_m(B) = \{\rho \in 2^m : \exists i_0 < \dots < i_{m-1} \in I : \mathbf{a}_{i_j} \in B \iff \rho(j) = 1\}$.

If $\mu \in 2^k$, $\eta \in 2^l$ and $l \leq k$, write $\eta \trianglelefteq \mu$ if η is a subsequence of μ , meaning that for some order preserving embedding $f : l \rightarrow k$ (where k and l are regarded as ordinals) $\forall i \in l, \mu(f(i)) = \eta(i)$.

Lemma 3.2. If $\langle \mathbf{a}_i \rangle_{i \in I}$ is a sequence of indiscernibles and $(I, <)$ is a complete and dense linear order without endpoints and $B \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$, is such that $\mathbb{G}(B) = \eta$ for some $\eta \in 2^{d+1}$ then

- (1) $|P_m(B)| = \binom{m}{\leq d}$
- (2) $\forall \rho \in 2^m, \rho \in P_m(B) \iff \eta \trianglelefteq \rho$.

Proof. The first claim is a corollary of Theorem 2.3. To see why, consider B , together with some $A \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ such that $|A| = m$. Theorem 2.3 describes what dichotomies on A arise as the endpoints of B are moved in an order-preserving fashion. But, dually, one can imagine that the endpoints of B are fixed, and that the points in A move. Claim (1) follows from the theorem, seen from the latter point of view. The second claim follows from the same theorem, together with an application of Proposition 2.2. \square

Assume that the formula $\varphi(\mathbf{x}, \mathbf{y})$ is NIP in the following lemma.

Lemma 3.3 (Transfer Lemma). *If $\langle \mathbf{a}_i \rangle_{i \in I}$ is a sequence of indiscernibles and $(I, <)$ is a complete and dense linear order without endpoints and $B \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ is defined by $\varphi(\langle \mathbf{a}_i \rangle_{i \in I}, \mathbf{c})$ and $A \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ is finite and $A' \subseteq A$ can be traced as $A' = A \cap B'$ where $B' \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ is such that $\mathbb{G}(B') = \mathbb{G}(B)$, then for some $\mathbf{c}' \in M^{|y|}$, $\varphi(A, \mathbf{c}') = A'$.*

Proof. Let $m = |A|$, and suppose $\mathbf{a}_{i_0} < \dots < \mathbf{a}_{i_{m-1}}$ is an enumeration of A . Since $\mathbb{G}(B') = \mathbb{G}(B)$, we have $P_m(B) = P_m(B')$ by Lemma 3.2 part (2). Let $\rho \in P_m(B')$ be such that for each $j = 0, \dots, m-1$, $\mathbf{a}_{i_j} \in A' \iff \rho(j) = 1$. Then $\rho \in P_m(B)$, and for some $\mathbf{a}_{k_0} < \dots < \mathbf{a}_{k_{m-1}}$ in $\langle \mathbf{a}_i \rangle_{i \in I}$,

$$\mathfrak{M} \models \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{k_j}, \mathbf{c})^{\rho(j)}$$

and thus

$$\mathfrak{M} \models \exists \mathbf{y} \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{k_j}, \mathbf{y})^{\rho(j)}$$

Then, by indiscernibility,

$$\mathfrak{M} \models \exists \mathbf{y} \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{i_j}, \mathbf{y})^{\rho(j)}$$

and the witnessing \mathbf{c}' is the desired parameter. \square

In the proof of the following theorem, for $p \in S_\varphi(A)$, we identify p and $B = \{\mathbf{a} \in A : \varphi(\mathbf{x}, \mathbf{a}) \in p\}$ without further comment.

Theorem 3.4. *For any $\varphi(\mathbf{x}, \mathbf{y})$, $\text{VCm}(\varphi) = \text{VCd}_{\text{ind}}(\varphi)$.*

Proof. First note that we always have $\text{VCm}(\varphi) \geq 0$ and $\text{VCd}_{\text{ind}}(\varphi) \geq 0$.

Now suppose $\text{VCd}_{\text{ind}}(\varphi) \geq d$, for some positive $d \in \omega$. Let $0 < \epsilon < 1/2$. By compactness, Ramsey's theorem, and saturation of the monster model, there is some indiscernible sequence $\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$ such that $|S_\varphi(A)| \geq |A|^{d-\epsilon}$ for arbitrarily large finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$.

Claim: $\exists B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$ with $\text{lg}(\mathbb{G}(B)) = d+1$.

First we argue that there is $B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$ with $\text{lg}(\mathbb{G}(B)) \geq d+1$. Suppose, to the contrary, that $\forall B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$, we have $\text{lg}(\mathbb{G}(B)) \leq d$. Then a

counting argument gives $K \in \omega$ such that for every finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$, $|S_\varphi(A)| < K \cdot |A|^{d-1}$, violating the hypothesis on $\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$.

Second, note that if there is some $B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$, with $lg(\mathbb{G}(B)) \geq d+1$, then by compactness and saturation there is some $B' \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$, with $lg(\mathbb{G}(B')) = d+1$. This proves the claim.

Now take B as in the claim. It follows from the Transfer Lemma that on any finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$,

$$\{B' \cap A : B' \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}, \mathbb{G}(B') = \mathbb{G}(B)\} \in Tr(\varphi, M^{|Y|})$$

This implies, by Theorem 2.3, that φ traces arbitrarily large d -maximum set systems, and, by Lemma 3.1, $Tr(\varphi, M^{|Y|})$ contains an infinite d -maximum set system. Thus $VCm(\varphi) \geq d$.

To show the other direction, suppose $VCm(\varphi) \geq d$. By compactness, saturation, and Ramsey's theorem (or, alternatively, by Erdős-Rado) there is an infinite indiscernible sequence $A = \langle \mathbf{a}_i \rangle_{i \in \omega}$ on which φ traces a d -maximum set system. It follows from the definition of d -maximum that $S_\varphi(A)$ witnesses that $VCd_{ind}(\varphi) \geq d$. \square

It should be noted that Guingona and Hill prove that $VCd_{ind}(\varphi)$ is equal to several other invariants, among which $VCm(\varphi)$ may obviously be included.

An immediate corollary is the following.

Corollary 3.5. *For any formula $\varphi(\mathbf{x}, \mathbf{y})$, $VCd(\varphi) = VCd_{ind}(\varphi)$ if and only if φ traces an infinite d -maximum set system, where $d = VCd(\varphi)$.*

This condition is easier to use in practice than a direct appeal to a nonconstructive combinatorial principle such as the Ramsey or Erdős-Rado theorem. We give an algebraic example in the theory of real closed ordered fields (RCOF). Though we make efforts to be precise in the following statements, we are just considering a definable family that results from a polynomial inequality where the coefficients form the parameter set.

Without loss we work in \mathbb{R} . Let \mathbf{y} be a finite sequence of variables. For a given $m \in \omega$, let Y_m denote the set of monomials which occur in the general polynomial of degree m with variables \mathbf{y} . Let $S \subseteq Y_m$ be chosen, and put $d = |S|$. Let $\mathbb{R}[\mathbf{y}]_S$ denote the polynomials with real coefficients all of whose constituent monomials occur in S . Consider $\mathcal{C} = \{\text{pos}(p) : p \in \mathbb{R}[\mathbf{y}]_S\}$, where $\text{pos}(p) = \{\mathbf{a} \in \mathbb{R}^{|Y|} : p(\mathbf{a}) \geq 0\}$.

Such a \mathcal{C} can clearly be traced by some $\varphi(\mathbf{x}, \mathbf{y}) \in L_{ring}$, in $\mathbb{R} \models \text{RCOF}$. It is known (see Floyd [4], section 8.1) that for such a φ , we have $VCm(\varphi) \geq d$. As it is well-known that $VCd(\varphi) = d$, we have $VCd_{ind}(\varphi) = VCd(\varphi)$ for polynomial inequalities φ .

Floyd's result is based on an application of Dudley's theorem ([3], Theorem 4.2.1), which can apply to somewhat more general situations (see Ben-David and Litman [2]).

Many familiar geometric families, such as circles, ellipses, half-spaces, hyperbolas, etc., therefore have VC_{ind} -density equal to VC-density. The

above approach notably does not apply to geometric families which are not polynomial definable (in the above sense) such as axis-parallel rectangles, or convex d -gons, though such results may follow from other means.

3.2. Stability. Here we show how to characterize the stability of φ using the maximum systems in $Tr(\varphi, M^{|y|})$. For a review of the relevant notions from stability theory see [6].

Recall that the *ladder dimension* of a formula $\varphi(\mathbf{x}, \mathbf{y})$ is defined by writing $LD(\varphi) \geq n$ if and only if there are $\mathbf{a}_0, \dots, \mathbf{a}_{n-1}$ in $M^{|\mathbf{x}|}$ and $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ in $M^{|\mathbf{y}|}$ such that $\mathfrak{M} \models \varphi(\mathbf{a}_i, \mathbf{b}_j) \iff i < j$. Finite ladder dimension is equivalent to stability for formulas. The VC-dimension can be thought of as a generalization of ladder dimension, and in general $\text{coVC}(\varphi) \leq LD(\varphi)$, where $\text{coVC}(\varphi)$ denotes the VC-dimension of $S_\varphi(M^{|y|})$ conceived as a set family.

For a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, define a graph $\mathcal{G}_{\mathcal{C}} = (V, E)$ where $V = \mathcal{C}$ and $E(C_1, C_2)$ holds if and only if $|C_1 \Delta C_2| = 1$. For $C_1, C_2 \in \mathcal{C}$, define $\text{dist}_h(C_1, C_2)$ to be the Hamming distance $|C_1 \Delta C_2|$, and $\text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2)$ to be the graph distance in $\mathcal{G}_{\mathcal{C}}$, with $\text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2) = \infty$ if C_1 and C_2 belong to different components.

The following was proved by Warmuth and Kuzmin [10], (Lemma 14).

Lemma 3.6. *Let X be a finite set. Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ is d -maximum, and $C_1, C_2 \in \mathcal{C}$. Then*

$$\text{dist}_h(C_1, C_2) = \text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2)$$

In particular, $\mathcal{G}_{\mathcal{C}}$ is connected.

The equivalence of dist_h and $\text{dist}_{\mathcal{G}_{\mathcal{C}}}$ is clearly still true when X is infinite, though the graph $\mathcal{G}_{\mathcal{C}}$ will not be connected in general. In fact, in many natural maximum set systems (for example, open intervals on a densely ordered set), $\mathcal{G}_{\mathcal{C}}$ is totally disconnected.

Theorem 3.7. *$\varphi(\mathbf{x}, \mathbf{y})$ is a stable formula with $LD(\varphi) \leq n$ if and only if for every $\mathcal{C} \in Tr(\varphi, M^{|y|})$ which is d -maximum for some $d \in \omega$, for any $C_1, C_2 \in \mathcal{C}$, $|C_1 \setminus C_2| \leq n$.*

Proof. First suppose $\varphi(\mathbf{x}, \mathbf{y})$ is a stable formula with $LD(\varphi) \leq n$.

Let $C_1, C_2 \in \mathcal{C}$, where $\mathcal{C} \in Tr(\varphi, M^{|y|})$ is d -maximum for some $d \in \omega$. We show $|C_1 \setminus C_2| \leq n$.

Let $A \subseteq C_1 \setminus C_2$ where $A = \{a_1, \dots, a_k\}$. By Lemma 3.6, after possibly reordering A , there are B_k, B_{k-1}, \dots, B_1 in \mathcal{C} , on the path from C_1 to C_2 in $\mathcal{G}_{\mathcal{C}|_A}$, such that $a_i \in B_j$ iff $i < j$. Thus $k \leq n$, and consequently $|C_1 \setminus C_2| \leq n$.

Conversely, suppose $LD(\varphi) > n$. Let $A = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subseteq M^{|\mathbf{x}|}$ and $B = \{\mathbf{b}_0, \dots, \mathbf{b}_n\} \subseteq M^{|\mathbf{y}|}$ such that $\mathfrak{M} \models \varphi(\mathbf{a}_i, \mathbf{b}_j) \iff i < j$. Put $\mathcal{C} = S_\varphi(B)|_A$. Then by Theorem 2.3, \mathcal{C} is 1-maximum, and $|B \setminus \emptyset| = |B| = n + 1$. □

The above theorem shows that much of the nature of Shelah's famous characterization of stable formulas by indiscernibles (see [15]) is already visible at the level of maximum traces. Unspooling the theorem reveals a structural characterization of stable maximum set systems, as we now show.

If X is a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$, define the *ladder dimension* of \mathcal{C} to be the maximal $n \in \omega$ such that there are $x_1, \dots, x_n \in X$ and $B_1, \dots, B_n \in \mathcal{C}$ with $x_i \in B_j$ if and only if $i < j$. Say that \mathcal{C} is *stable* just in case it has finite ladder dimension.

If $A \subseteq X$, define $\mathcal{C}\Delta A = \{C\Delta A : C \in \mathcal{C}\}$.

Lemma 3.8. *If $\mathcal{C} \subseteq \mathcal{P}(X)$ has $LD(\mathcal{C}) = n$, then for any $A \subseteq X$, we have $LD(\mathcal{C}\Delta A) \leq 2n$, and this bound is tight.*

Proof. For brevity, we show only the tightness of the bound. Fix $n \in \omega$. Let X be the integers between $-n$ and n , inclusive, but not including zero. Let $\mathcal{C} = \{[0, i] \cap X : 0 < i \leq n\} \cup \{[-i, 0] \cap X : 0 < i \leq n\}$. Clearly $LD(\mathcal{C}) = n$. But $LD(\mathcal{C}\Delta[-n, -1]) = 2n$. \square

Note that the example in the above proof is 1-maximum.

Corollary 3.9. *Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be d -maximum of ladder dimension n .*

- (1) *If $\emptyset \in \mathcal{C}$, then $\mathcal{C} \subseteq [X]^{\leq n}$.*
- (2) *$\mathcal{C}\Delta B \subseteq [X]^{\leq 2n}$ for any $B \in \mathcal{C}$.*
- (3) *$\mathcal{C} \subseteq [X]^{\leq 2n}\Delta B$ for any $B \in \mathcal{C}$.*

Proof. The claim in (1) is clear from Theorem 3.7. For the claim in (2) note that $\emptyset \in \mathcal{C}\Delta B$ (because $B \in \mathcal{C}$), and the ladder dimension of $\mathcal{C}\Delta B$ is at most $2n$ by Lemma 3.8. Therefore $\mathcal{C}\Delta B \subseteq [X]^{\leq 2n}$ by (1). Claim (3) follows after applying ΔB to both sides of the containment in (2). \square

Clearly every finite maximum set system is stable, and so the above says something about the structure of finite maximum classes.

The $2n$ bound in Corollary 3.9 part (2) is tight, as the following example shows. Let $X = \{1, \dots, 2n\}$, and $\mathcal{C} = \{D : D \in [X]^{\leq n}\}$. Clearly $LD(\mathcal{C}) = n$. Now putting $B = \{1, \dots, n\}$ gives $\mathcal{C}\Delta B$ an element of cardinality $2n$.

On the other hand, it seems possible that for some $B \subseteq X$, it may be the case that $\mathcal{C}\Delta B \subseteq [X]^{\leq n}$, where the hypotheses are as in Corollary 3.9. However, since the hypotheses admit all finite maximum classes, this conjecture may be too optimistic. Such a result would clearly be the best possible.

It is also evident from the above that the stable maximum set systems are exactly those maximum set systems $\mathcal{C} \subseteq \mathcal{P}(X)$ which can be realized as $\mathcal{C} \subseteq [X]^m\Delta B$ for some $m \in \omega$ and $B \subseteq X$ (because the latter systems are clearly stable).

It would be useful to know whether every \mathcal{C} of ladder dimension n embeds into a $O(n)$ -maximum set system \mathcal{C}' . See [2] for relevant embedding notions. If this were true, it would have interesting applications to definability of types and VC-density.

4. CONCLUSION

In model theory, much of the combinatorial content comes from considering formulas restricted to indiscernible sequences. The existence of sequences of indiscernibles is guaranteed by Ramsey’s theorem (and compactness), though it is rarely required to exhibit a concrete sequence of indiscernibles.

When dealing with a certain formula $\varphi(\mathbf{x}, \mathbf{y})$ on a sequence $A = \langle \mathbf{a}_i \rangle_{i \in I}$ it is a weaker condition to assume that φ is maximum on A than to assume that A is indiscernible. However, as we have seen, if φ is maximum on A , that provides “enough” indiscernibility for some combinatorial notions to come through. Namely, dp -rank, NIP, and stability can all be understood in terms of the maximum property.

Unlike indiscernible sequences, maximum domains are frequently easily constructible. In the semilinear case, it follows from the work of Floyd and Dudley that a basic semilinear family will be maximum on a set of points which is in “general position,” for which it is sufficient to take a randomly selected set of points.

Moreover, the similarity of maximum families and formulas on indiscernible sequences provides a point of contact between work done in computational learning theory and model theory, where, especially recently, researchers are pursuing compatible combinatorial goals, but without a common framework.

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